Test 1 — Solutions

Problem 1. Let ABC be a triangle, let O be its circumcentre, let A' be the orthogonal projection of A on the line BC, and let X be a point on the open ray AA' emanating from A. The internal bisectrix of the angle BAC meets the circumcircle of ABC again at D. Let M be the midpoint of the segment DX. The line through O and parallel to the line AD meets the line DX at N. Prove that the angles BAM and CAN are equal.

Solution 1. Choose a point Y such that $AONY$ is a parallelogram. Since the lines AD and ON are parallel, this point lies on the line AD (see Fig. 1). We prove that the triangles AOY and AXD are similar. Since the line AN bisects the segment OY the conclusion follows.

It is well known that the internal bisectrix AD of the angle ABC is also the internal bisectrix of the angle OAA'. Next, the corresponding sides of the triangles OND and ADX are parallel, so these triangles are similar. Hence $ON/OD = AD/AX$. Since $OD = OA$ and $ON = AY$, this shows that $AY/AO = AD/AX$. Along with the equality of the angles OAY and DAX, this proves the required similarity of the triangles AOY and AXD.

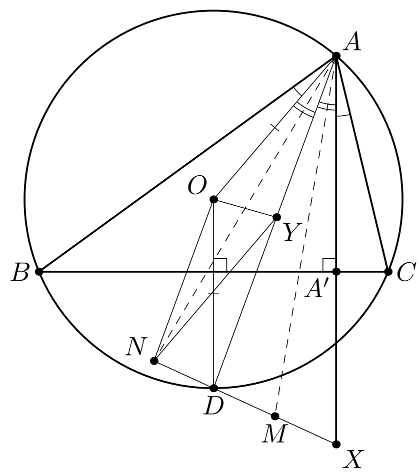

Fig. 1

Solution 2. Since the angle BAC is internally bisected by AD, it is sufficient to prove that so is the angle MAN.

Let P, Q, R, S be the points of intersection of the pairs of lines AM and OD, OA and XD, AN and OD, and AD and QR, respectively (see Fig. 2). Since the angles MAN and PAR are the same, we show that AD is the internal bisectrix of the latter.

Apply Menelaus' theorem to both triangles DMP and DRS and the transversal AOQ to write

$$\frac{AM}{AP} \cdot \frac{OP}{OD} \cdot \frac{QD}{QM} = 1$$

and

$$\frac{AD}{AS} \cdot \frac{OR}{OD} \cdot \frac{QS}{QR} = 1,$$

respectively. Since $OD \parallel AX$ and $DM = MX$, we have $AM = MP$. In triangle AQD, the line ON is parallel to AD, so R lies on its median from Q, and hence $AS = SD$. Thus $MS \parallel PD$, which yields $\frac{QM}{QD} = \frac{QS}{QR}$. Combining the obtained relations we get

$$\frac{OP}{OD} = \frac{QM}{QD} \cdot \frac{AP}{AM} = \frac{QS}{QR} \cdot \frac{AD}{AS} = \frac{OD}{OR},$$

or $OD^2 = OP \cdot OR$. Thus, $OA^2 = OP \cdot OR$. This shows that the triangles OAR and OPA are similar, and $\angle OAR = \angle OPA$. Finally, by $OA = OD$ we obtain

$$\angle RAD = \angle OAD - \angle OAR = \angle ODA - \angle OPA = \angle DAP,$$

as required.

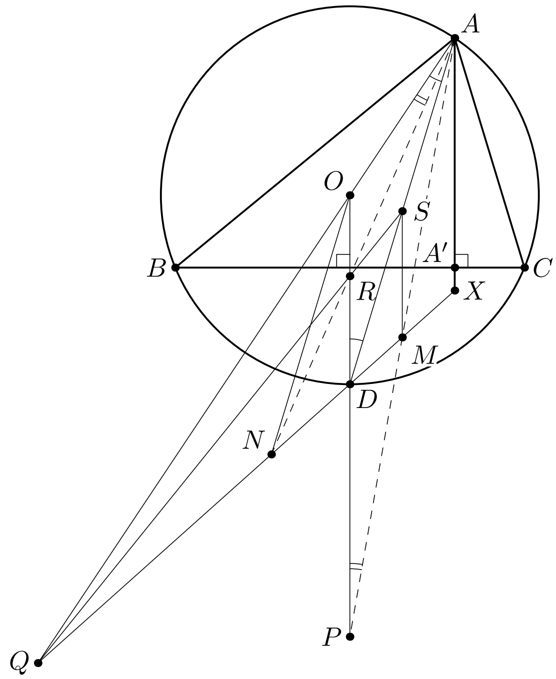

Fig. 2

Remark. The conclusion is that AM and AN, and AB and AC are pairs of isogonal lines. This is still true if A separates A' and X, but in this case AD is the external bisectrix of the angle MAN, and the angles BAM and CAN are supplementary.

Problem 2. Let ABC be a triangle, and let r denote its inradius. Let R_A denote the radius of the circle internally tangent at A to the circle ABC and tangent to the line BC; the radii R_B and R_C are defined similarly. Show that $1/R_A + 1/R_B + 1/R_C \le 2/r$.

Solution. We shall prove that $1/R_A = (a/\Delta)\cos^2(B/2 - C/2)$, where Δ denotes the area of the triangle ABC. Similar formulae hold for R_B and R_C, and the conclusion follows at once; in addition, this shows that equality holds if and only if the triangle ABC is equilateral.

To prove the above formula for R_A, let γ_A be the circle tangent at A to the circle ABC and tangent at T to the line BC, assume the triangle ABC has unit circumradius, and invert from A with unit power. In what follows, X' will denote the image of the point $X \ne A$ under this inversion.

Under this inversion, the line BC is transformed into a circle $AB'C'$ centred at some point Ω; the circle ABC is transformed into the line $B'C'$; and γ_A is transformed into a line ℓ through T' and parallel to $B'C'$.

Let D be the orthogonal projection of A on the line BC. Then $AD' = 1/AD = 1/h_A$, where h_A is the length of the altitude from A in the triangle ABC, and $\Omega T' = \Omega A = 1/(2h_A)$.

Next, let A_1 be the antipode of A in γ_A, so A_1' is the orthogonal projection of A on ℓ, and $AA_1' = 1/AA_1 = 1/(2R_A)$.

Finally, let O denote the circumcentre of the triangle ABC and notice that the angles OAD and $\Omega AA_1'$ are both congruent to the absolute value of the difference of the internal angles of the triangle ABC at B and C, to obtain

$$\cos(B-C) = \frac{AA_1' - \Omega T'}{\Omega A} = \frac{\frac{1}{2R_A} - \frac{1}{2h_A}}{\frac{1}{2h_A}} = \frac{h_A}{R_A} - 1 = \frac{2\Delta}{aR_A} - 1,$$

whence the desired formula via obvious standard transformations.

Remarks. (1) Instead of the inversion from A, we could equally well have considered a homothecy centred at A transforming the circle ABC into γ_A.

(2) We may also consider the circles externally tangent at A, B, C, respectively, to the circle ABC, and tangent to the lines BC, CA, AB, respectively. Letting R_A', R_B', R_C' denote their radii, the corresponding inequality now reads $1/R_A' + 1/R_B' + 1/R_C' < 1/(2r)$. Notice that if the triangle ABC is isosceles, say $AB = AC$, then the circle corresponding to the apex A degenerates into the parallel through A to BC, so $R_A' = \infty$ and $1/R_A' = 0$, and the inequality is still valid.

Problem 3. A *Pythagorean triple* is a solution of the equation $x^2 + y^2 = z^2$ in positive integers such that $x < y$. Given any non-negative integer n, show that some positive integer appears in precisely n distinct Pythagorean triples.

Solution 1. We show by induction $n \geq 0$, that 2^{n+1} appears in precisely n distinct Pythagorean triples. Since no Pythagorean triple contains 2, the assertion holds for $n = 0$. For the induction step, let $n \geq 1$, and assume that 2^n appears in exactly $n-1$ distinct Pythagorean triples. The latter produce $n-1$ distinct non-primitive Pythagorean triples each containing 2^{n+1}. To conclude the proof, we show that 2^{n+1} appears exactly once in a primitive Pythagorean triple. Recall that the primitive Pythagorean triples are described by the well-known formulae $x = v^2 - u^2$, $y = 2uv$, $z = u^2 + v^2$, where u and v are coprime positive integers, not both odd, and $u < v$. Since x and z are both odd, if 2^{n+1} appears in the triple, then $2^{n+1} = y = 2uv$, and since $u < v$ and u and v have opposite parity, necessarily $u = 1$ and $v = 2^n$. Consequently, 2^{n+1} appears in exactly n distinct Pythagorean triples.

Solution 2. If $P(m)$ is the number of Pythagorean triples containing the positive integer m, and if $P_0(m)$ is the number of primitive such triples, then $P(m) = \sum_{d|m} P_0(d)$. Since $P_0(1) = P_0(2) = 0$ and $P_0(2^k) = 1$, $k \geq 2$ (as in the previous solution), it follows that $P(2^{n+1}) = n$, so 2^{n+1} appears in exactly n distinct Pythagorean triples.

Solution 3. We show that if p is a prime congruent to 3 modulo 4, then p^n appears in exactly n Pythagorean triples, and is moreover always the smallest entry of any such.

Since p is congruent to 3 modulo 4, -1 is a quadratic non-residue modulo p, so no power of p can be the largest entry of a Pythagorean triple. Hence, if p^n is a member of a Pythagorean triple, then $p^{2n} = b^2 - a^2$ for some positive integers $a < b$, so $b - a = p^k$ and $b + a = p^{2n-k}$ for some non-negative integer $k < n$. Clearly, every such k corresponds to a solution and there are precisely n distinct Pythagorean triples containing p^n, namely,

$$p^n, \quad p^k(p^{2(n-k)} - 1)/2, \quad p^k(p^{2(n-k)} + 1)/2, \quad k = 0, 1, \ldots, n-1.$$

It is worth noticing that this argument avoids appealing to the parametric representation of Pythagorean triples.

Problem 4. Let k be a positive integer congruent to 1 modulo 4 which is not a perfect square, and let $a = (1 + \sqrt{k})/2$. Show that $\{\lfloor a^2 n \rfloor - \lfloor a \lfloor an \rfloor \rfloor : n = 1, 2, 3, \ldots\} = \{1, \ldots, \lfloor a \rfloor\}$.

Solution. Let $a_n = an - \lfloor an \rfloor$, $n = 1, 2, 3, \ldots$. Since $a^2 = a + (k-1)/4$, it follows that $\lfloor a^2 n \rfloor = \lfloor an \rfloor + n(k-1)/4$, and $(a-1)\lfloor an \rfloor = (a-1)(an - a_n) = n(k-1)/4 - (a-1)a_n$, so, adding $\lfloor an \rfloor$ to each side, $a\lfloor an \rfloor = \lfloor an \rfloor + n(k-1)/4 - (a-1)a_n = \lfloor a^2 n \rfloor - (a-1)a_n$. Since a is irrational, the a_n form a dense subset of the open unit interval $(0,1)$, so, by the preceding, the differences $\lfloor a^2 n \rfloor - a\lfloor an \rfloor = (a-1)a_n$ form a dense subset of the open interval $(0, a-1)$. Finally, since $\lfloor a^2 n \rfloor - \lfloor a \lfloor an \rfloor \rfloor = \lceil \lfloor a^2 n \rfloor - a \lfloor an \rfloor \rceil = \lceil (a-1)a_n \rceil$, the conclusion follows.

Problem 5. Given an integer $N \geq 4$, determine the largest value the sum

$$\sum_{i=1}^{\lfloor k/2 \rfloor + 1} \left(\lfloor n_i/2 \rfloor + 1 \right)$$

may achieve, where k, n_1, …, n_k run through the integers subject to $k \geq 3$, $n_1 \geq \cdots \geq n_k \geq 1$, and $n_1 + \cdots + n_k = N$.

Solution. The required maximum is $2\lfloor N/3 \rfloor + \epsilon$, where $\epsilon = 1$ if N is divisible by 3, and $\epsilon = 2$ otherwise.

For more convenience, given a list of k real numbers, the sublist consisting of the $1 + \lfloor k/2 \rfloor$ largest entries will be referred to as the *upper half* of the list, and its complement, i.e., the sublist consisting of the $\lfloor (k+1)/2 \rfloor - 1$ smallest entries, as the *lower half* of the list. Notice that the lower half of a list consisting of at least three real numbers is never empty.

To maximise the sum s in the statement, we list a sequence of operations which transform any given partition of N into at least three positive integers into another such whose lower half is all 1, and the upper half is all 2 except possibly one unit entry; moreover, each operation yields a partition into at least three positive integers, and does not decrease s, whence the conclusion. In what follows, n_1, …, n_k will denote a generic partition of N into at least three positive integers; the obvious verifications are omitted.

If the number of unit entries in the partition is less than $\lfloor (k+1)/2 \rfloor - 1$, i.e., the lower half has some entry $n_i > 1$, splitting n_i into 1 and $n_i - 1$ increases length by 1, and s by at least 1 if k is odd, and preserves it otherwise; in either case, s does not decrease.

If the number of unit entries in the partition exceeds $\lfloor (k+1)/2 \rfloor$, i.e., the upper half has at least two unit entries, replacing two 1's by one 2 increases s by 1 if k is odd, and preserves it otherwise; in either case, s does not decrease, and since $N > 3$ the resulting partition has length at least three. (In fact, the length of the resulting partition would be less than three only in case $N = 3$, and the partition we start with is 1, 1, 1 — the unique partition of 3 into three positive integers. This is, however, ruled out by hypothesis.)

Consequently, a partition of N into at least three positive integers can be transformed into another such whose lower half is all 1, and the upper half has at most one unit entry; moreover, s does not decrease in the process, and the lengths of the partitions involved are at least three. Henceforth, all partitions are assumed to have such a structure.

If the upper half has no unit entry, but has some odd entry $n_i > 1$, splitting n_i into 1 and $n_i - 1$ increases length by 1, and s by 1 if k is odd, and preserves it otherwise; in either case, s does not decrease, and the outcome is a partition into at least three positive integers, whose lower half is all 1, and the upper half has exactly one unit entry and fewer odd entries exceeding 1.

If the upper half has exactly one unit entry and some odd entry $n_i > 1$, replacing that unit entry and n_i by 2 and $n_i - 1$ preserves length, increases s by 1, and the resulting partition has length at least three, an all 1 lower half, and the upper half has fewer odd entries exceeding 1 and no unit entry.

Consequently, every partition of N into at least three positive integers can be transformed into another such with an all 1 lower half, and an all even upper half except possibly one unit entry;

moreover, at each stage, the length of the partition is at least three, and s does not decrease. Henceforth, all partitions are assumed to have such a structure.

If the upper half has no unit entry, but has some entry $n_i > 2$, splitting n_i into 1, 1 and $n_i - 2$ increases length by 2, preserves s and yields a partition into at least three positive integers, whose lower half is all 1, and the upper half is all even except for exactly one unit entry and has fewer entries exceeding 2.

Finally, if the upper half is all even except for exactly one unit entry, and has some entry $n_i > 2$, splitting n_i into 2 and $n_i - 2$ increases length by 1, and s by 1 if k is odd, and preserves it otherwise; in either case, s does not decrease, and the outcome is a partition of length at least three, whose lower half is all 1, and the upper half is all even with fewer entries exceeding 2.

Consequently, any given partition of N into at least three positive integers can be transformed into another such whose lower half is all 1, and the upper half is all 2 except for at most one unit entry; moreover, the transformation does not decrease s, and all partitions have length at least three. For this 'standard' partition, it is readily checked that $s = 2\lfloor N/3 \rfloor + \epsilon$, where $\epsilon = 1$ if N is divisible by 3, and $\epsilon = 2$ otherwise. The conclusion follows.

Remark. Maximising partitions are not necessarily unique. For instance, if m is an integer greater than 1, then

$$\underbrace{2, \ldots, 2}_{m+1}, \underbrace{1, \ldots, 1}_{m} \quad \text{and} \quad 4, \underbrace{2, \ldots, 2}_{m-1}, \underbrace{1, \ldots, 1}_{m}.$$

are both maximising partitions of $3m + 2$ into at least three positive integers; the former is 'standard', whereas the latter is not. Similarly, if $m > 2$, then

$$\underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{m} \quad \text{and} \quad 4, \underbrace{2, \ldots, 2}_{m-1}, \underbrace{1, \ldots, 1}_{m-2}.$$

are both maximising partitions of $3m$ into at least three positive integers; again, the former is 'standard', whereas the latter is not.